Elementary and middle school students’ analyses of pictorial growth patterns

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A B S T R A C T
Research has suggested the importance of incorporating algebraic thinking early and throughout the K-12 mathematics curriculum. One approach to help children develop algebraic reasoning is through the examination of pictorial growth patterns, which serve as a context for exploring generalization. The purpose of this study was to compare how elementary and middle school students analyze pictorial patterns, with a focus on whether students used figural or numerical reasoning. Task-based interviews were conducted with a second grader, fifth grader, and eighth grader in which they were asked to describe, extend, and generalize two pictorial growth patterns. Using a phenomenographic approach, analyses showed younger students used figural reasoning more than older students, but all students did not exclusively use figural or numerical reasoning. The students’ generalizations included informal notation, descriptive words, and formal notation. The findings suggest that pictorial growth patterns are a promising tool for young students’ development of algebraic thinking.

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1. Introduction

Over the past 20 years, the topic of algebra in school curricula and standards documents has become increasingly emphasized as a primary content strand across K-12 mathematics (e.g., Australian Curriculum, Assessment, and Reporting Authority, 2010; National Council of Teachers of Mathematics, 2000). Traditionally, algebra has been considered a formal course taken in late middle or early high school. As recent as the late 1980s, algebra was a course deemed abstract in nature and only appropriate for college-bound students who were developmentally ready (Garmoran, 1987). In today’s global economy, algebra is considered the “gatekeeper” course to higher-level mathematics and in turn, a college education. Increasingly, algebra is becoming the focus of policymakers, researchers, and practitioners with a movement toward all students taking a formal course in algebra in eighth grade (National Mathematics Advisory Panel, 2008).

Regardless of when students take algebra, students need opportunities throughout their elementary and middle school years to develop their abilities to generalize, the core of algebraic thinking (Kaput, 1999). The recently released Common Core State Standards in Mathematics (CCSS-M) in the United States outline that elementary school students should be generalizing about our number system and about other patterns in their world (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). One example is the analysis of geometric patterns (Grade 4, CCSS-M). Geometric or pictorial growth patterns serve as a rich context for exploring generalization. The purpose of this study was to explore and compare how elementary and middle school students analyze pictorial growth patterns.

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2. Background

2.1. Curriculum and standards

The Curriculum and Evaluation Standards for School Mathematics (1989) and the Principles and Standards for School Mathematics (2000) by the National Council of Teachers of Mathematics (NCTM) in the United States called for new thoughts about algebraic thinking to be integrated into work in elementary school mathematics. The Principles and Standards state:

By viewing algebra as a strand in the curriculum from pre-kindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school. For example, systematic experience with patterns can build up to an understanding of the idea of function, and experience with numbers and their properties lays a foundation for later work with symbols and algebraic expressions (p. 37)

As outlined, NCTM proposes integrating algebraic thinking early and throughout elementary school to promote students’ abilities to make generalizations. The idea of early algebra “encompasses algebraic reasoning and algebra-related instruction among young learners – from approximately 6 to 12 years of age” (Carraher & Schliemann, 2007, p. 670). More recently, the CCSS-M further support the development of algebraic thinking in elementary school students.

2.2. Describing algebra and defining algebraic reasoning

Since there is a push to integrate algebraic reasoning across the grade levels in K–12 mathematics, describing algebra and defining algebraic reasoning are necessary. First, algebra includes generalized arithmetic, a problem-solving tool, the study of functions, and modeling (Bednarz, Kieran, & Lee, 1996). Carraher and Schliemann (2007) define algebraic reasoning as “psychological processes involved in solving problems that mathematicians can easily express using algebraic notation” (p. 670). Algebra is typically thought of as a study of symbol systems while algebraic thinking is often used to indicate the kinds of generalizing that come before or during the use of formal algebraic notation (Smith, 2003). NCTM’s Principles and Standards (2000) explain that algebraic thinking focuses on analyzing change, generalizing relationships among quantities, and representing these mathematical relationships in various ways.

2.3. Pictorial growth patterns

Making algebra accessible for all students is important for two reasons: the competitiveness of nations in our increasingly global economy and providing access to higher-salary career opportunities for minority students (Moses & Cobb, 2002; NCTM, 2000; National Math Advisory Panel [NMAP], 2008; RAND Mathematics Study Panel, 2003). Asking students to analyze contexts and pictures offers opportunities for more students to experience success in algebraic thinking well before they are enrolled in a formal algebra course. Thus, implementing activities in elementary and middle school to promote algebraic thinking is critical. One of many approaches to help young children generalize about patterns is through the examination of pictorial growth patterns (Ferrini-Mundy, Lappan, & Phillips, 1997; Friel & Markworth, 2009; Orton, Orton, & Roper, 1999). Sometimes called geometric patterns, a pictorial growth pattern is a sequence of figures that change in a predictable fashion from one figure to the next (Billings, 2008). The goal is for students to analyze, describe, and extend the pattern and ultimately, generalize about relationships in the patterns. The patterns provide a context for students to make generalizations, and the nature of the patterns allows for a variety of approaches to making generalizations.

2.4. Theoretical framework

The overarching theoretical framework that guided the development of this study’s tasks is the construct of figural concepts, objects that have both spatial properties and conceptual properties (Fischbein, 1993). In other words, objects in the tasks (see Appendix A) have figural cues because they have been constructed in a certain way spatially. For example, the square tiles have been arranged in a square shape in the first task, and the consecutive figures relate to each other. They also have a structural property so students can generalize about a concept embedded in the figure. For example, in the second task, the structure of the figures allows students to make connections to area of a rectangle. The theory of figural concepts provides a lens for considering the modes of reasoning students might use to approach the tasks.

Two modes of reasoning are often used when analyzing pictorial growth patterns, figural and numerical. “A numerical mode of inductive reasoning uses algebraic concepts and operations (such as finite differences), whereas a figural mode relies on relationships that could be drawn visually from a given set of particular instances” (Rivera & Becker, 2005, p. 199). Often, teachers think of a procedure of manipulating numbers to develop a formula rather than using the picture to generate the formula. Therefore, teachers tend to model numerical reasoning to their students while ignoring the characteristics of the picture pattern.

When analyzing pictorial growth patterns, children often use a “covariation analysis of the pattern” (Smith, 2003) in which they focus on the changes from one figure in the pattern to the next. They use the previous figure to build up to a new figure and identify what stays the same and what changes in the pattern. This can also be called recursive induction.
When asked to generalize to figures later in the sequence, children may switch to a “correspondence analysis” (Smith, 2003) in which they are able to describe the relationship between the figure number and the changing aspect of the dependent variable. Further, they generalize about what any figure in the pattern will look like. When generalizing about any figure in a sequence, students are sometimes able to provide a closed or explicit formula.

2.5. Existing research

In a study involving pictorial growth patterns, Rivera and Becker (2005) found that twenty-six of forty-two prospective elementary and middle school teachers used predominantly numerical reasoning rather than figural reasoning to analyze patterns. Seventeen of the twenty-six teachers who used a numerical strategy made generalizations about patterns using a covariation analysis of the pattern or recursion. They used the previous figure in a sequence to determine the next one. The researchers found that the students who used more figural than numerical reasoning were better able to explain and justify the closed or explicit formulas they developed through correspondence analysis.

Beyond looking at prospective teachers, Becker and Rivera (2005) conducted a 5-year study in a school district assessing ninth graders’ abilities to analyze patterns and functions. Over the period of 5 years, they assessed close to 60,000 students. They used pictorial growth patterns and found that 70 percent of the participants could extend patterns one at a time, but only 15 percent could develop a closed, explicit formula through generalizing algebraically. To follow up this study, Rivera (2007) conducted interviews of 22 ninth graders in which he asked them to analyze pictorial growth patterns. He analyzed the strategies of the students to find that most students who used only numerical reasoning and developed a closed formula were not able to explain and justify their formulas.

In their work with second and fourth graders, Moss, Beatty, McNab, and Eisenband (2006) worked with students on integrating figural and numerical patterns. They facilitated a series of lessons to promote students translating between the two representations of patterns. After the treatment, they found students who participated in the lessons were stronger on number knowledge and on making generalizations with explicit reasoning than students who did not receive the lesson instruction. Students who used both numerical and figural reasoning were better able to make generalizations, develop closed formulas, and make connections to what the terms in the formula mean.

Like the aforementioned research, other studies have focused on algebraic thinking of same-grade level or same-age samples of students (Carraher, Martinez, & Schliemann, 2008; Markworth, 2010; Rivera, 2010; Tanisli, 2011). There is a gap in the literature on the examination of wider age/grade level range of students. The current study fills that gap by providing insights about the similarities and differences of primary, upper elementary, and middle school students’ algebraic thinking about figural concepts in the form of pictorial growth patterns. In an era of standards (e.g., CCSS-M) and reports (e.g., NMAP) outlining the importance of algebraic thinking in elementary school, there is an increasing need to learn more about how children across grade levels think about and approach algebraic tasks.

2.6. Research questions

The purpose of this study was to explore how students of various ages and grade levels engage in algebraic reasoning when presented with pictorial growth patterns during task-based interviews, through a phenomenographic research design. Specifically, the research questions were:

- What are the processes and strategies a second grader, fifth grader, and eighth grader utilize to describe and extend pictorial growth patterns?
- What are the processes and strategies a second grader, fifth grader, and eighth grader utilize to generalize about pictorial growth patterns?
- What are the similarities and differences in the employed strategies of a second grader, fifth grader, and eighth grader in their analyses of pictorial growth patterns?

3. Method

3.1. Phenomenography

Phenomenography (Akerlind, 2005; Marton & Booth, 1997) is a qualitative research methodology that aims to describe the variation in the ways people experience a situation or phenomenon. It is utilized to describe aspects of a situation that are critically different within a group involved in that situation. The data is interrogated for commonalities and differences in the emerging themes. Analysis is an iterative process; the researcher repeatedly returns to the data to refine the themes. Phenomenography has been used in the context of students’ responses to mathematical tasks (e.g., Herbert & Pierce, 2013). In this study, phenomenography is the approach for comparing how the participants think about the mathematics in the tasks.
3.2. Participants

Three students of the following grade levels volunteered to participate in this study: second grade, fifth grade, and eighth grade. The sampling strategy was purposeful and convenience, in that order. In phenomenography, the goal is to have variation in the sample rather than large numbers of people. Since I was interested in having a wide variation in ages/grade levels, I purposefully sought children of particular ages; I asked children of friends for convenience. I will refer to the participants as Dara, Jenny, and David, respectively.

Dara is 7 years old and in the second grade at an elementary school in a college town nestled in a larger metropolitan area of a southeastern state. The school she attends is approximately 50% White, 20% Black, 16% Hispanic, and 34% receiving free/reduced lunch. Dara described her mathematics time in second grade as starting with a problem of the day. She described specific topics or types of problems her teacher presents: money, measurement, “word problem,” and operations with base ten block representations. She also talked about playing games during math on Fridays. Dara knew our interview related to patterns. However, without prompting by me, she shared what she had learned about patterns when she described her math class. She discussed growing patterns and laid square tiles on the table in a long row in the following sequence: one green, two red, three yellow, two green, three red, four yellow. She described how each color was getting “older.” In summary, Dara said her teacher “makes stuff fun” in math. Dara’s parents expressed that Dara does not exhibit the same high level of confidence in mathematics as she does in reading and writing. Despite this lower level of confidence, she has been successful in school mathematics.

Jenny, 11 years old, attends fifth grade at an elementary school located in a rural area of a mid-Atlantic state. The school is 85% White, 10% Black, and 26% receiving free/reduced lunch. Jenny is enrolled in an advanced mathematics class as they are learning sixth grade concepts as outlined by state standards. When asked about her mathematics class, Jenny said “we’re somewhat doing the basics of algebra” with no prompting about the topic of algebra from me. When asked to elaborate, she said they use variables. She described her sixth grade math as mostly review with only some “new things.” She said class usually starts with a message on the board, “turn to this page in your textbook.” Her teacher usually says “what we’re going to learn about and he explains it. And, then we take notes, and he helps us on worksheets and stuff.” Jenny shared that she enjoys math class, but she is not particularly fond of fifth grade.

David is 14 years old and attends eighth grade at the only middle school in the same rural district as the elementary school that Jenny attends. The middle school is 82% White, 15% Black, and 24% receiving free/reduced lunch. David is enrolled in Geometry, the high school course, as an eighth grader. He completed Algebra I when he was in the seventh grade. David expressed that he likes math. He described his Algebra I and Geometry classes to be exactly the same routine. He shared, “a typical day is taking notes with examples that look a lot like what we’re doing on our homework and then we do our homework.”

3.3. Data collection

In phenomenography, data collection typically includes semi-structured interviews with a small sample. Task-based interviews are semi-structured interviews in which participants explain and justify their thinking during and after solving a mathematics problem. Many researchers in mathematics education have used task-based interviews (e.g., Goldin, DeBellis, DeWindt-King, Passantino, & Zang, 1993; Tanisli, 2011) to examine mathematical thinking. Piaget introduced task-based interviews in the 1970s, but the method has evolved to include open-ended prompting and structured think-aloud protocols (Clement, 2000). Goldin (2000) asserts that “task-based interviews can serve as research instruments for making systematic observations in the psychology of learning mathematics and solving mathematical problems” (p. 520). There is evidence that task-based interviews provide insights into the participants’ knowledge, problem-solving techniques, and reasoning (Newell & Simon, 1972; Schoenfeld, 2002).

3.4. Procedure

The participants engaged in a task-based interview for approximately 30 min. The interviews occurred at a site convenient for the participants. During this time, the participants were asked to analyze two pictorial growth patterns presented in Appendix A (Smith, Silver, & Stein, 2005). The task-based interviews were semi-structured. Participants were presented with one picture in each pattern at a time. Participants were first asked to describe the picture. Then, they had to explain the next picture in the pattern, reason about later entries in the pattern (e.g., picture #20, picture #100), and generalize about any entry in the pattern. During the interviews, participants had access to plain white paper, pencils, the papers with the pictorial growth pattern, and square tiles. I videotaped their work by zooming in on the workspace. This allowed me to capture their voices as they explained their thinking. Further, I was able to review how they wrote their work, used square tiles, and pointed to pictures or other written work to explain their reasoning.

3.5. Data analysis process

The interviews were video recorded for transcription and for follow-up analyses. Both the video recorded interviews and the transcriptions underwent iterative manual analyses. There are important steps to the phenomenographic approach to
Table 1
A depiction of analysis of select data using phenomenography.

<table>
<thead>
<tr>
<th>Observations: pictorial growth pattern #1</th>
<th>Categories</th>
<th>Dimensions</th>
<th>Outcome space or themes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dara EXTEND</td>
<td>Counts up to figure out how many squares on each side using tally marks to skip pictures in between</td>
<td>Physical features</td>
<td>Figural reasoning</td>
</tr>
<tr>
<td></td>
<td>Looks at picture number to figure out how many squares on each side</td>
<td>Building up</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Uses the picture number, skips a number, and gets the number of squares on each side</td>
<td>Physical features</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Draws informal notation for her trick of skipping the number</td>
<td>Correspondence analysis</td>
<td></td>
</tr>
<tr>
<td>Jenny EXTEND</td>
<td>Says the pattern is plus one because height and base gets bigger by one</td>
<td>Covariation analysis</td>
<td>Numerical reasoning</td>
</tr>
<tr>
<td></td>
<td>Subtracts to find out how many numbers in between and adds to highest known picture (4)</td>
<td>Numerical features</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Notices quantity increases four each time</td>
<td>Generalization in words</td>
<td>Figural reasoning</td>
</tr>
<tr>
<td>David EXTEND</td>
<td>Writes a formula based on symbolic numbers</td>
<td>Numerical features</td>
<td>Generalization – words</td>
</tr>
<tr>
<td></td>
<td>Interprets formula in terms of quantity in original picture, not in terms of spatial features</td>
<td>Correspondence analysis</td>
<td>Numerical reasoning</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Generalization through explicit formula</td>
<td>Generalization – notation</td>
</tr>
</tbody>
</table>

Qualitative analysis. First, I watched the videos and read the transcriptions. I interpreted the responses and recorded observations about each part of the interview. Then, I grouped the observations into themes called categories. Next, I described the categories in my notes, grouped them together, and proposed possible dimensions. I revisited the data to further refine the categories and dimensions. As I examined the dimensions, I thought about how they were linked and the hierarchical relationships of the dimensions. Finally, I generated what is called an “outcome space” (Marton & Booth, 1997), which includes essentially the broad statements about how the students as a collective unit approached the mathematical tasks. The iterative process of generating observations (also called “meaning statements”), categories, dimensions, and the outcome space provided the structure for a systematic approach to the data analysis. Table 1 displays a depiction of this analysis process for a select sample of the data.

The data and findings are reported in two parts. First, each participant’s responses to the tasks are described thoroughly followed by a summary. Second, the communalities and differences across the three participants’ responses are discussed. Comparisons are made between the three students as to the type(s) of reasoning, figural or numerical, each used as he/she approached the figural concept tasks.

3.6. Validity

Validity was attained through checking for representativeness, debriefing with a peer, triangulating across data sources, and weighing the evidence. To ensure external and internal validity, the task-based interviews included member checking. This was done throughout the interviews as I asked participants to explain or elaborate on their responses. I asked them to restate some responses or asked for further clarification if I did not understand. Also, I had to consider how knowing the children and conducting the interviews myself could influence my analysis; therefore, I debriefed with a peer multiple times during the analysis to limit bias. Since I triangulated across interviews, I was also able to examine if researcher effects were apparent. To confirm my themes, I searched for disconfirming evidence and attempted to make meaning of outliers.

4. Dara, the second grader

4.1. Dara – task #1

When Dara, a second grader, was presented with picture #1 for the first task and asked to describe what she sees, she immediately said, “3, 6, 9, 12.” When asked to elaborate, she circled four groups of three (see Fig. 1), double circling the
corner squares. Subsequently, when she was asked to compare picture #1 to #2, she said, “I can say they can still be in the same way.”

Dara proceeded to circle four groups of four and said, “4, 8, whatever.” She said, “they’re like twins, only this one (referring to picture #2) is four more (as she circles the four corner pieces).” Dara later says, “4, 8, 12, 16” by counting on from eight to obtain twelve, then counting on from twelve to get sixteen.

When presented with picture #3, Dara says she just noticed how this could be a tool for helping students with counting and says, “5, 10, 15, 20.” Again, she circles the four groups of five like she did on pictures #1 and #2. Dara refers to her classmates and says, “they could count by threes, they could count by fours, they could count by fives, it would be a great tool in our classroom.” Making a prediction about what picture #4 will look like, Dara said “probably sixes” and draws the picture (see Fig. 2). When asked about picture #5, Dara said it would be “sevens” and “a little bit bigger,” but she wanted to “try that out.” She proceeded to draw a square shape with seven small squares on each side.

Next, I asked Dara to predict what picture #10 would look like, Dara used tally marks to count up and said, “12.” When I asked her to explain, she said, “I used tally marks to know it was five more.” She then counted above the tally marks and said, “8, 9, 10, 11, 12.” (see Fig. 3). I asked her what she meant by 12, and Dara drew a large square made up of twelve smaller squares on each side. Dara proceeded to explain the relationship between picture #10 having 12 squares “in each row.” She says 12 represents the “number in each row” and 10 represents the “picture.” She explains her numerical notation (see Fig. 4) by saying “you start with picture 1, you skip a number, and you come up to 3.” Similarly, she uses the same notation to describe pictures #2, #3, #4, #5, and #10.

Next, I asked Dara to predict picture #25. Dara applied the same strategy as illustrated in Fig. 4. She started at 25, skipped 26, and said there would be 27 squares in each row of the square. In reference to the squares in the corner, she recognized they overlap. She says, “even these that overlap, it doesn’t matter. It’s still a row.” She knows she is double counting the
Fig. 3. Dara’s tally mark and counting method for determining picture #10 will have 12 squares on each side.

Fig. 4. Dara’s notation for describing the pattern. For picture #1, she says to skip 2 to find 3 squares are in each row. For picture #2, she says to skip 3 to find 4 squares in each row. For picture #4, skip 5 to find 6 squares in each row. Likewise, for picture #5, skip 6 to find 7 squares in each row.

Fig. 5. Dara’s strategy to determine 102 squares in each row for picture #100.

corner squares, but she recognizes this does not change the description of her pattern. To describe her strategy again, she says, “the picture is the first number you want to start with. Then, you skip a number and you come up with the row number.” When asked to consider picture #100, Dara immediately applies the same strategy, shown in Fig. 5. She says there will be 102 squares in each row.

I asked Dara to describe her strategy for finding picture #100, and she started to generalize on her own using invented symbolic notation. She said, “start, to skip, to answer.” Her symbol for “start” was a circle with a vertical line. For “skip,” she used a backwards “C.” For “answer,” she sketched a circle with a horizontal line (see Fig. 6).

She explained her rationale for having notation by saying, “if you have like one hundred, um, one thousand, four hundred, ninety-three, I don’t think you could just pop that out of your mind, the next number and the next number after that, if you’re in second grade. So, it’s easier to do that (referring to her notation and strategy).” When I asked Dara how to find any picture in the pattern, she immediately went to a specific example of picture #7. However, she then explained, “to figure out the row, I’d give you the picture # and you’d try to come up with that (referring to the number in each row).” (see Fig. 7) She said, “All you have to do is skipcount. Some people may not understand, but once you do it, poof, you know it.”

4.2. Dara – task #2

The interview with Dara on the second task was much shorter. When I asked her to describe what she saw in picture #1, she wrote the number sentence “1 + 1 = 2” to represent the picture. Next, I asked her to compare picture #1 to picture #2.

Fig. 6. Dara’s notation for her strategy of “start, skip, answer.”

She explained that she saw a “main idea” in picture #1 that appeared in picture #2 (see Fig. 8). Subsequently, she notices the “main idea from picture #2 appears in picture #3.” Dara describes an “S” shape in the pattern. After showing her the first four pictures, I ask her to predict what the fifth picture would look like. She says she would not be able to predict what it would look like. At this point, I ended the interview, knowing the task was very difficult, perhaps not developmentally appropriate for Dara as a second grader.

4.3. Summary of Dara’s strategies

Dara’s strategy for describing the pattern in the first task, extending the pattern, and generalizing about the pattern can be summarized as figural reasoning. While she does “skipcount” to determine the number of squares in any picture, her rationale for this strategy is based on the relationship between the picture number and the number of squares in each “row” or side of the square. Therefore, her rule is embedded in figural reasoning. She uses her understanding of numbers or numerical reasoning to complete the “skipcounting,” but her strategy is focused on the picture. She makes a clear connection between the picture number and the “number in each row,” demonstrating correspondence analysis of change.

5. Jenny, the fifth grader

5.1. Jenny – task #1

For the first task, Jenny, a fifth grader, describes picture #1 as squares making up a square. She also mentions using an array she learned in third grade. She said she could multiply three times three to get nine and subtract the missing square to obtain eight total squares. When asked to compare picture #2 to picture #1, she notices there are three squares in the base and height in picture #1 versus four squares in the base and height for picture #2. In turn, Jenny noticed picture #3 is bigger. When asked to predict what picture #4 would look like, Jenny said, “it’s going to be a six height and a six base because it gets bigger by one.” She continued, “it’s like an in-out thing, like input-output. You have to figure out the pattern, and the pattern in that is plus one.” When asked to determine picture #10, Jenny says there will be twelve squares on each side “because there are six numbers in between ten and four. And, on picture #4, there are six and there’s six in between so you do six plus six and you get twelve.”

To determine if Jenny would apply a similar strategy on another picture, I asked her to describe picture #25. She explained, “I just subtracted twenty-five minus four to find out how many numbers are in between four and twenty-five. It would be twenty-seven squares because I found out there were twenty-one numbers in between. So, it would be six plus twenty-one, and I get twenty-seven. The six comes from six squares in that (points to the bottom row of picture #4).”

Jenny’s strategy for generalizing is not dependent on the picture number given. Instead, she uses information about the largest picture she has (#4) to determine any other picture in the pattern. To investigate how Jenny will approach later pictures in the pattern, I ask her about picture #100. Jenny thinks quietly, then writes 96 + 6 on her paper to determine 102.
She says, “102 squares on the bottom, on the height, on the top, and on the other side.” When I asked her how she obtained her answer, she said, “because I subtracted 100 minus 4 and got 96 so there are 96 numbers in between. Then, you already have 6, so you plus 6 plus 96 and you get 102.” When I ask her about any picture in the pattern, she says, “draw it out?” So, I ask again for any number in the pattern such as 144. Since I mentioned an exact number, she includes that number in her description for any number. She says, “you take the number and subtract it from the highest picture you have, like 144 minus 4. Then, you add that to six in this case, so 146.”

5.2. Jenny – task #2

For the second picture task, Jenny immediately sees two squares in picture #1 that “if you added two squares, it would make a big square.” When comparing picture #2 to picture #1, Jenny sees, “they added three on” (see Fig. 9). As she looks at picture #3, she confuses herself as she talks about individual squares that are added, but then, she says, “it added another row on top. There was three going up, now, there’s four going up. It’s kind of grown taller and fatter.”

As Jenny looks at picture #4, she says, “oh my goodness, they added a lot again.” As she continues to analyze, she says, “there’s four on the bottom and four on the top. And, it’s three wide, it was two wide on that one (points to picture #3) and one wide on that one (pointing to picture #2). So, they’re making it longer on the top and bottom, and fatter in the middle.” Next, I asked Jenny to extend the pattern to picture #5. After drawing picture #5 (see Fig. 10), Jenny explains what she did, “I added one to make it fatter, and I added one to make it longer and taller.” I asked her to elaborate and she says, “I made it one more to make it fatter this way (draws line that is red in Fig. 10), and I made it one more taller this way (draws line that is blue in Fig. 10).”

As Jenny makes observations, she describes the shape as being “kind of like an S.” I ask her to tell me what picture #10 will look like. She says, “it will have five more this way going across and five more on top.” So, I prompted her to make statements about the rows. She says, “it will have 10 in the bottom row, 10 in the top row, and this way, the height will be

Fig. 9. Jenny circled the squares she saw as “added” moving from picture #1 to #2.

Fig. 10. Jenny’s picture #5 with indicators for “fatter” (red) and “taller” (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)
11. It will be 11 tall and 9 wide in the rows in between.” To describe picture #25, Jenny says, “if you compare it to number 5, it will have 20 more going this way (across) and 20 more going this way (height).” I prompted her, and she said it would be 25 in the top row, 25 in the bottom row, 26 in the height, and 24 going across in the rows between the top and bottom. Without prompting about each row, Jenny is able to describe picture #100 in the same way. She explains how she obtains 99 squares for each row in the middle by “subtracting 100 minus 1.” She says it will be “101 tall because she saw the pattern in each picture.”

To ask Jenny to generalize about any number, Jenny said, “the number of the picture will be the number that’s on the bottom and the top. And, then, if you take the number of the picture and plus one, you’ll get the height. And, then you take the picture number and minus it by one, and you will get the fatness of it.”

5.3. Summary of Jenny’s strategies

Jenny’s strategies link numerical and figural reasoning. In the first task, her strategy of using the largest picture she knows to figure out any picture depends upon the figural characteristics of the picture and depends upon a covariation analysis of change. She uses numerical reasoning to determine “how many numbers are in between” when she subtracts four from the desired picture number. She describes her final answer in terms of the number of tiles in the base, height, etc. Again, she combines her numerical and figural reasoning. The nature of the second task promotes purely figural reasoning in Jenny. She describes the figures using vocabulary words “taller” and “fatter” and describes any picture indicating correspondence analysis of change.

6. David – the eighth grader

6.1. David – task #1

When David was asked to describe picture #1 in the first task, he said, “it’s made up of eight tiles, it’s a square. it forms a perimeter instead of an area, it’s not like a filled in figure.” David describes pictures #2 and #3 in a similar way, and I ask him to predict what picture #4 will look like: “A square that only forms a border made up of 6, 6, 4, and 4, made up of 20 tiles.” I asked him to elaborate on his thoughts. He draws picture #4 and explains how he divided the tiles into groups of six, six, four, and four (see Fig. 11).

As we move to picture #5, David says, “I’m noticing a pattern that it’s four more tiles each time.” To move beyond consecutive pictures, I asked David about picture #10. David says, “It would be the tenth picture in the sequence. The first one was eight. So, it would be eight plus four for every picture after that. So, you could come up with a formula.” David proceeds to write the formula (see Fig. 12).

His description of his formula follows: “the number of tiles, t, equals eight plus four times the number there is minus one.” When I asked David for more elaboration on his formula:

Fig. 11. David’s drawing of picture #4 with groups of six, six, four, and four.

Fig. 12. David’s “formula” for the picture pattern in the first task.
D: So, there’s eight in the first one, so you know you’re always going to have that eight. Then, four more in picture two.
T: Why are there four more?
D: 8 and then 12, four more.
T: Okay.
D: So, then, the number that it is minus one would put it at one, so four times one is four.
T: What does the $n$ represent in your formula?
D: The number that it is in the pattern.
T: The number that it is in the pattern. What number?
D: Picture 1, Picture 2, Picture 3

To further his explanation, David proceeds to show is work for picture #10 in the pattern (see Fig. 13). After determining the 44 tiles for picture #10, David says, “so picture #10 would have 44 tiles, would be a square, and would only form a border.” To explain picture #100, David again applied his formula to obtain 404 tiles. He said, “picture #100 would be a square, form only a border, and have 404 tiles.”

6.2. David – task #2

For the second pictorial growth pattern, David notices two tiles “in a diagonal arrangement” for picture #1. When asked to compare picture #2 to picture #1, he notices the top square from picture #1 has “been translated up one unit and to the right one unit and a line of three has been put in between them.” For picture #3, he sees the same square being shifted “up one and over again. Then, one was added to the line of three and then four more.” When I ask David what picture #4 will look like, he says, the same one in the upper right corner will be “shifted up one and over again. Two will be added to the top of the lines of four, and a line of five will be added.” In Fig. 14, he shows how the upper right square in picture #1 is translated up and over each time through his shading and diagonal line.

Next, I asked David what picture #10 will look like. He sits quietly for a few seconds and says, “okay, if I can come up with a formula for this picture (referring to his picture #4).” David proceeds to say as he writes his formula (see Fig. 15), “Tiles would be the original two, then the number that it is in the sequence plus one to get that height, two plus one (points to height of picture #2), three plus one (points to height of picture #3), four plus one (points to height of picture #4), and it would be multiplication, and then to get that (points to width of rectangle in middle of picture #4), the number that it is minus one.” He explains, picture #4 as being the width of the rectangle, four minus one equals three, times the height of the rectangle, four plus one equals five, to get the fifteen squares in the rectangle. Then, he adds the two “original squares” on the sides. He proves that it works for picture #4 through applying his formula (see Fig. 15).

When asked to make predictions about picture #10, he says there will be eleven horizontal rows and nine vertical rows. He explains that would be 99 tiles plus the two original tiles equals 101 tiles. To generalize, he says, “If you take the picture number and add one, that’s then number of horizontal rows. The picture number minus one is the number of vertical rows. Multiply those two numbers, then add the two original squares.”

![Fig. 13. David’s work for determining the number of tiles in the picture #10.](image)

![Fig. 14. David’s drawing of picture #4 with the square being translated “up and over one” shaded.](image)
6.3. Summary of David’s strategies

David employs more numerical reasoning in the first task and figural reasoning in the second task. He is definitely focused on the number of tiles in each picture as a function of the picture number, but his “formula” in the second task is based on what he sees in the pattern. David uses a constant in both formulas as the number of squares in picture #1 of each pattern. In the first task, his answer is focused on the total number of tiles, a numerical representation. In the second task, the link between his formula and the picture is more apparent, and he is able to explain the formula and number of tiles in terms of the pictures. In both tasks, he uses a correspondence analysis of change.

7. Findings

The preceding descriptions outline in detail how the three students of different ages described, extended, and generalized about two pictorial growth patterns. In this section, I present the findings by addressing the three research questions. Rather than answer each research question individually, the questions will be addressed as a composite unit by focusing on the two themes that emerged in the outcome space. Through examining the strategies the three students used to describe, extend, and generalize about pictorial growth patterns, an automatic comparison of such strategies occurred. The analysis resulted in two interrelating themes as the outcome space, which are conceptually organized in Fig. 16. The two themes are: an intersection between figural and numerical reasoning and making generalizations using symbols and/or words. The two themes and how they are organized are outlined below.

Fig. 15. David applies his formula for picture #4 to prove that it works. The “2” is the original squares from picture #1. \((n + 1)\) and \((n - 1)\) are the dimensions of the rectangle.

Fig. 16. A conceptual framework of the findings.
Table 2
Students’ approaches to the tasks at three stages.

<table>
<thead>
<tr>
<th>Student name</th>
<th>Pictorial growth pattern #1</th>
<th>Pictorial growth pattern #2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Describe</td>
<td>Extend</td>
</tr>
<tr>
<td>Dara</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>Jenny</td>
<td>F</td>
<td>N/F</td>
</tr>
<tr>
<td>David[a]</td>
<td>F</td>
<td>N</td>
</tr>
</tbody>
</table>

F = figural reasoning, N = numerical reasoning, X = did not complete.

[a] The first one listed is the first type of reasoning evident in the student’s thinking.

7.1. Theme #1: the intersection between figural and numerical reasoning

The three students in the study did not use one type of reasoning exclusively, and they used both types of reasoning to make generalizations about the patterns. Further, by combining figural and numerical reasons, the students were able to make generalizations about the pictorial growth patterns, hence the reason “generalizations” is the center or heart of the Venn diagram. In the cases of Dara, Jenny, and David, they would not have made generalizations without examining the spatial features of the pictures and applying their knowledge of number relationships. Table 2 shows whether each student used figural or numerical reasoning at each step of the progression of describing, extending, and generalizing about the patterns. In the cases where two types of reasoning are listed such as “F/N,” this indicates the student used primarily figural reasoning initially combined with some numerical reasoning later.

Examples for each student will be discussed to show how they each used a combination of figural and numerical reasoning as they completed the tasks. Dara’s reasoning on the first task is primarily based on figural reasoning. She focuses on how many squares are “in each row.” To relate the picture number to the number of squares in each row, she uses her ability to reason numerically to describe her skip counting technique. Jenny’s strategy for describing the first pattern is based on the figure as she notes the number of squares in the base and the height of each square. When asked to extend and generalize about the pattern, Jenny draws more on her numerical reasoning than figural reasoning. Recall that to determine picture #25, Jenny determined “how many numbers are between” 4 and 25 by subtracting. Then, she added the difference, the number of squares in the base of picture #4. While her reasoning is still dependent upon the figure, her strategy is focused on what she knows about numbers. David’s strategy for describing the first pattern is focused on the figural characteristics of the pattern. When he moves to extending, he primarily uses numerical reasoning as his explanations center around the increase of four tiles from one picture in the sequence to the next. When David writes a closed, explicit formula for any picture in the pattern, David primarily uses numerical reasoning to explain. The only reference to figural reasoning is his understanding of the “8” in his formula as the original eight tiles in the first picture.

7.2. Theme #2: making generalizations using notation and/or descriptive words

Making generalizations using notation and/or descriptive words was common among the three case studies. As indicated in Fig. 16, the generalizations result from the use of numerical and figural reasoning. The types of generalizations the students use are descriptive words or notation. The notation is invented or formal in nature. Examples for each student will be discussed to show how they all use notation and/or descriptive words to generalize the patterns.

For task one, Dara uses descriptive words to describe how she determines how many square tiles are in each row. She explains her “start, skip, answer” strategy as starting with the picture number, skipping the next number, and the answer is the next number. She repeatedly references “skip counting” as she talks. As she is describing her generalization, she uses invented notation for the start number, skip number, and answer. Her notation is a circle with a vertical line, a backwards “C,” and a circle with a horizontal line, respectively. She defends her invented notation by saying it’s “really very easy.” She said you only need to skip count and “poof, you know it.” Dara is able to generalize in words, but she sees the need to represent her words with her own invented notation.

Jenny uses descriptive words to describe her generalization of the both patterns. For the second pattern, she focuses on the “fatness” and “tallness” of the picture. She is able to explain the height as one more than the picture number and the “fatness” as one less than the picture number. She also explains the top and bottom rows have the same number of squares as the picture number. Jenny does not use notation on either of her generalizations, but she still exhibits algebraic reasoning through her descriptions.

David uses formal, algebraic notation on both tasks to write closed, explicit formulas. For the second task, he writes the formula as \(2 + (n + 1) \times (n - 1)\). David is able to explain in words why his formula works. He explains the number of horizontal rows of squares is the picture number plus one \((n + 1)\), and the number of vertical rows of squares is the picture number minus one \((n - 1)\). He is able to explain why you multiply these two together to get the area of the rectangle in the middle. He refers to the two squares on the ends as the reason to add two. David is able to generalize with a formula, explain his
formula, articulate how to find the number of squares in any number in the pattern, and describe how any number in the pattern will look.

7.3. Summary: progression of strategies

When analyzing the three students’ strategies, there is a progression of strategies students appear to employ as they develop mathematically, age naturally, and experience patterns in their mathematics classrooms and in the world around them. First, the students engage in describing patterns based on number and geometric relationships such as counting, area, multiplication, etc. Then, students make generalized statements about patterns with words or informal notation. All three students progressed through these two phases as they analyzed the patterns. Finally, some students generalize using notation and explain their notation in terms of the pattern. Dara generalized with informal notation while David was the only student to generalize with formal algebraic notation. Since David is in eighth grade and has received formal algebra instruction, this is not a surprise. Despite the differences in strategies, the abilities to make generalizations about the patterns were strikingly similar, regardless of age.

8. Discussion and implications

There are three primary findings from this study. First, students did not use figural or numerical reasoning exclusively. The figural concepts, presented as pictorial growth patterns, appeared to promote both types of reasoning as the students described, extended, and generalized the patterns. This supports the work of former researchers who found students use a combination of numerical and figural reasoning, although one type may be the dominant choice for a student (Becker & Rivera, 2005).

Second, the younger students were more apt to rely more heavily on figural reasoning when analyzing pictorial growth patterns; they paid more attention to the structural and physical features. Both Dara and Jenny used some level of figural reasoning for every stage (describe, extend, generalize) they were able to complete. This is developmentally appropriate in a learning trajectory of algebraic reasoning. While David used figural reasoning for the second task, he was very focused on the number of tiles in both tasks and paid less attention to the concept embedded in the figure (figural concept). It is plausible that the difficulty of the second task prompted David to rely more on figural reasoning to determine a generalization. This raises the need to incorporate pictorial growth patterns of various levels of challenge for students. In the cases of these three students, their choice of reasoning (figural, numerical, or both) is likely affected by the instructional experiences they have experienced in their mathematics classrooms.

Third, students generalized using various ways to communicate – informal notation, oral language, and formal notation. Dara created her own notation system to describe how to determine the number of squares on each side of the first pictorial pattern. Jenny used language such as “fatter” and “taller” as she generalized about the second pictorial pattern. David used formal algebraic notation as he generalized about both patterns. Developmentally, the progression of generalizations among students is expected, although oral language may precede informal notation for younger students. The variety of representations used to communicate is likely impacted by these students’ experiences in school mathematics.

As a result of this research study, there are several implications for mathematics teaching. It appears pictorial patterns would serve as valuable tools in elementary and middle school classrooms to promote algebraic reasoning. When students make sense of their generalizations in terms of a pictorial context, students will likely be able to describe relationships among mathematical objects, the heart of algebraic thinking. Presumably, then, they would be able to make meaning of formulas rather than just manipulating numbers. Their work with generalizing the patterns can potentially connect to understanding the constant and slope in linear equations.

As evident in the findings, providing picture patterns seems to produce several different ideas about the same pattern. In this study, Dara, Jenny, and David shared different generalizations for the first task. In a classroom, this promotes student solution sharing. Then, students will likely make connections between equivalent expressions and formulas.

Dara, Jenny, and David’s interviews are indicative that students can generalize about pictorial growth patterns. As teachers of mathematics, encouraging students to describe their generalization in terms of the picture is important. This promotes the connection between numerical and figural reasoning, a significant one in algebraic thinking. As outlined by Fischbein (1993), the concept is embedded in the figure. When students abandon the picture and look exclusively at the numbers, teachers should provide prompts and coach the students to examine the pictures.

Incorporating the analyses of pictorial growth patterns into elementary and middle school mathematics promotes the use of figural and numerical reasoning together. Most importantly, students use algebraic thinking to generalize about the relationships. Understanding mathematical relationships and generalizing about those relationships are key parts of the critical foundation for students’ success in their formal algebra courses, the gatekeeper to opportunities.
Appendix A.

Pictorial pattern for task #1

Picture 1

Picture 2

Pictorial pattern for task #2

Picture 3

Picture 4

Picture 5